Počinjemo u 14^{15}

5. (**6 bodova**) Primjenom Laplaceove transformacije riješite donju diferencijalnu jednadžbu s početnim uvjetom.

$$\mathcal{L} \quad \begin{cases} y''(t) + 5y'(t) + 4y(t) = u(t) & f(t) \\ \boxed{y(0) = 0}, \quad \boxed{y'(0) = 1.} \end{cases}$$

$u(t) \circ\!\!-\!\!\bullet \frac{1}{s}$

$y(t) \circ\!\!-\!\!\bullet Y(s)$

$y'(t) \circ\!\!-\!\!\bullet sY(s) - y(0) = sY(s)$

$y''(t) \circ\!\!-\!\!\bullet s \cdot sY(s) - y'(0) = s^2 Y(s) - 1$

$s^2 Y(s) - 1 + 5sY(s) + 4Y(s) = \frac{1}{s}$

$Y(s)\left[s^2 + 5s + 4\right] = \frac{1}{s} + 1$

$Y(s) = \frac{1}{s^2+5s+4} \cdot \frac{1+s}{s}$

$\frac{s+4s}{s(s+1)+4(s+1)}$

$(s+4)(s+1)$

$3s^2 + 15s + 12 =$

$= 3\left[1s^2 + 5s + 4\right]$

$3\left[(s+1)(s+4)\right]$

$s^2 + 5s + 4 = (s-s_1)(s-s_2)$

$(s+1)(s+4)$

$s^2 + 5s + 4 = 0$

$\leadsto s_1 = -1$

$\leadsto s_2 = -4$

$$Y(s) = \frac{\cancel{1+s}}{s(s+4)\cancel{(s+1)}} = \frac{1}{s(s+4)}$$ razstav na parc. razlomke $= \frac{A}{s} + \frac{B}{s+4}$

$$\frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4} \quad \Big| \cdot s(s+4)$$

$$e^{at}u(t) \circ\!\!-\!\!\circ \frac{1}{s-a}$$

$$1 = A(s+4) + Bs$$

$$0 \cdot s^2 + 0 \cdot s + 1 = (A+B)s + 4A$$

$\langle 1 \rangle$: $1 = 4A \rightsquigarrow A = 1/4$

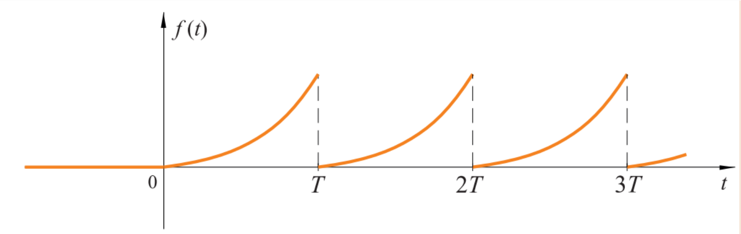
$\langle s \rangle$: $0 = A+B \rightsquigarrow B = -A = -1/4$

$$Y(s) = \frac{1}{s(s+4)} = \frac{1/4}{s} + \frac{-1/4}{s+4}$$

$$\frac{1}{4}u(t) \qquad -\frac{1}{4}e^{-4t}u(t)$$

$$\boxed{y(t) = \frac{1-e^{-4t}}{4}u(t)}$$

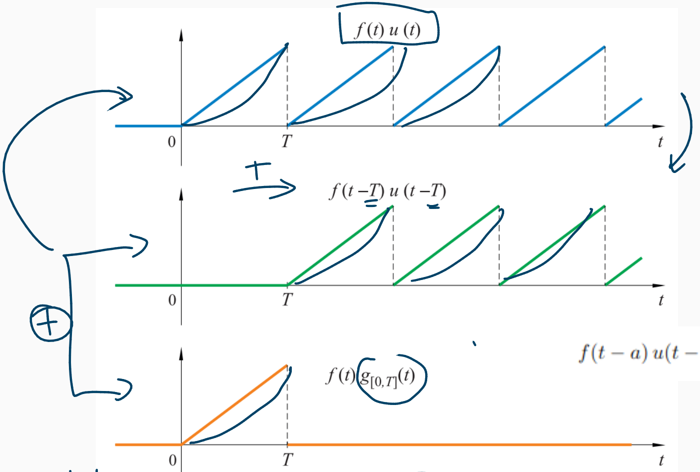
Laplaceov transformat periodičnih funkcija



Ako je $f(t)$ periodička funkcija perioda T. Onda vrijedi

$$f(t)\,u(t) \circ\!\!-\!\!\bullet\, \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t\,.$$

Izvod na predavanjima.



kašnjenje od T

$f(t-a)\,u(t-a) \circ\!\!-\!\!\bullet\; e^{-as}F(s)$

$\mathcal{L}\{f(t)u(t)\} = F(s)$

f periodička f-ja s periodom T

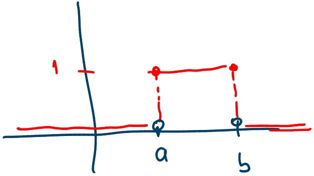
$$\mathcal{L}:\quad \underbrace{f(t)\,u(t)}_{F(s)} = \underbrace{f(t-T)\,u(t-T)}_{e^{-Ts}\cdot F(s)} + \underbrace{f(t)\,g_{[0,T]}(t)}_{\int_0^T f\cdot g_{[0,T]}\cdot e^{-st}\,dt}$$

$$F(s) = e^{-Ts} \cdot F(s) + \int_0^T f(t)\, e^{-st}\, dt$$

$$F(s)\left[1 - e^{-Ts}\right] = \int_0^T f(t)\, e^{-st}\, dt$$

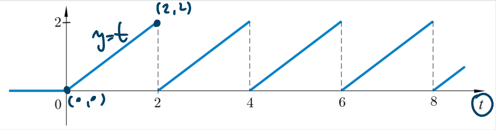
$$\underbrace{F(s) = \frac{1}{1 - e^{-Ts}} \int_0^T f(t)\, e^{-st}\, dt}$$

$$g_{[a,b]}(t) = \begin{cases} 1, & a \le t \le b \\ 0, & \text{inafe} \end{cases}$$



Primjer 29

Odredite Laplaceov transformat funkcije zadane donjim grafom.



$f(t)$

periodička s

periodom $T=2$

$\mathcal{L}(y''+y'=f(t)$

$$f(t) \circ\!\!-\!\!\bullet \frac{1}{1-e^{-2s}} \cdot \int_0^2 t \cdot e^{-st}\,dt = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-2s}-2se^{-2s}}{s^2}$$

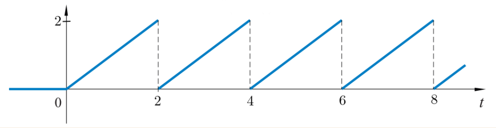
$$\int_0^2 \underbrace{t}_{u}\underbrace{e^{-st}dt}_{dv} = \left[\begin{matrix} u=t & e^{-st}dt=dv \\ du=dt & v=\dfrac{e^{-st}}{(-s)} \end{matrix}\right] = \frac{t\,e^{-st}}{(-s)}\Bigg|_0^2 - \int_0^2 \frac{e^{-st}}{(-s)}dt =$$

$$= \frac{2e^{-2s}-0}{-s} + \frac{1}{s}\int_0^2 e^{-st}dt = -\frac{2e^{-2s}}{s} + \frac{1}{s}\,\frac{e^{-st}}{(-s)}\Bigg|_0^2 =$$

$$= -\frac{2e^{-2s}}{s} - \frac{1}{s^2}\left(e^{-2s}-1\right) = \frac{1-e^{-2s}-2se^{-2s}}{s^2}$$

Primjer 29

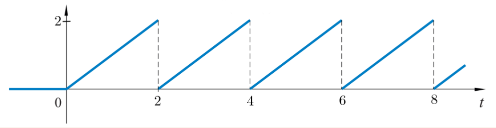
Odredite Laplaceov transformat funkcije zadane donjim grafom.

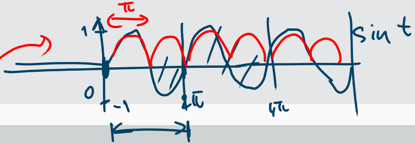


Rj: $\frac{1-e^{-2s}(2s+1)}{s^2(1-e^{-2s})}$

Primjer 29

Odredite Laplaceov transformat funkcije zadane donjim grafom.





Rj: $\frac{1-e^{-2s}(2s+1)}{s^2(1-e^{-2s})}$

Primjer 30 (*)

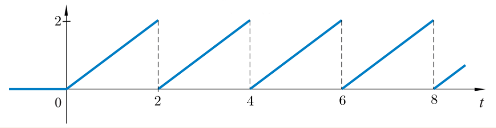
Odredite $\mathcal{L}\{|\sin t|\, u(t)\}$. ⇝ koji je period $|\sin t|$

$T = \pi$

...

Primjer 29

Odredite Laplaceov transformat funkcije zadane donjim grafom.



Rj: $\frac{1-e^{-2s}(2s+1)}{s^2(1-e^{-2s})}$

Primjer 30 (*)

Odredite $\mathcal{L}\{|\sin t|\, u(t)\}$. **Rj:** $|\sin t|\, u(t) \circ\!\!-\!\!\!-\!\!\bullet \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})}$

Inverzna transformacija

Primjer 31

Riješi diferencijalnu jednadžbu s početnim uvjetom :

$$\mathcal{L} \, / \quad \begin{cases} f''(t) + f'(t) = \sin t + e^t, \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

$\boxed{?}$ $f(t) \circ\!\!-\!\!\bullet \; F(s)$

$f'(t) \circ\!\!-\!\!\bullet \; sF(s) - \overset{0}{f(0)}$

$f''(t) \circ\!\!-\!\!\bullet \; s^2F(s) - s\overset{0}{f(0)} - \overset{0}{f'(0)}$

$\sin t \circ\!\!-\!\!\bullet \; \frac{1}{s^2+1}$

$e^t \circ\!\!-\!\!\bullet \; \frac{1}{s-1}$

$$s^2F(s) + sF(s) = \frac{1}{s^2+1} + \frac{1}{s-1}$$

$$F(s)\left[s^2+s\right] = \frac{s\cancel{-1} + s^2 + \cancel{1}}{(s^2+1)(s-1)}$$

$$F(s)\,\cancel{\left[s^2+s\right]} = \frac{\cancel{s+s^2}}{(s^2+1)(s-1)} \quad \Big| : (s+s^2)$$

$$\boxed{F(s) = \frac{1}{(s^2+1)(s-1)}} \quad \bullet\!\!-\!\!\circ \quad f(t) = ??$$

$$\cancel{s^2+1=0}$$

$$F(s) = \frac{As+B}{s^2+1} + \frac{C}{s-1} = \frac{1}{(s^2+1)(s-1)} \quad \Big| \cdot (s^2+1)(s-1)$$

$$(As+B)(s-1) + C(s^2+1) = 1$$

$$As^2 + Bs - As - B + Cs^2 + C = 1$$

$$(A+C)s^2 + (B-A)s - B + C = 1 + 0s + 0s^2$$

$$\langle s^2 \rangle: \quad A + C = 0 \rightsquigarrow A = -C$$

$$\langle s \rangle: \quad B - A = 0 \rightsquigarrow B = A = -C \qquad \boxed{B = A = -1/2}$$

$$\langle 1 \rangle: \quad -B + C = 1 \rightsquigarrow C + C = 1 \Rightarrow \boxed{C = 1/2}$$

$$F(s) = \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{s-1} = \boxed{-\frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s-1}}$$

$$\mathcal{L}^{-1}$$

$$f(t) = \left[-\frac{1}{2}\cos t - \frac{1}{2}\sin t + \frac{1}{2}\cdot e^{t}\right]\cdot \cancel{u(t)}$$

Primjer 31

Riješi diferencijalnu jednadžbu s početnim uvjetom :

$$\begin{cases} f''(t) + f'(t) = \sin t + e^t, \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

Stavimo $f(t) \circ\!\!-\!\!\bullet F(s)$. Pa vrijedi:

Primjer 31

Riješi diferencijalnu jednadžbu s početnim uvjetom :

$$\begin{cases} f''(t) + f'(t) = \sin t + e^t, \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

Stavimo $f(t) \circ\!\!-\!\!\bullet F(s)$. Pa vrijedi:

$$f'(t) \circ\!\!-\!\!\bullet sF(s) - f(0) = sF(s)$$

Primjer 31

Riješi diferencijalnu jednadžbu s početnim uvjetom :

$$\begin{cases} f''(t) + f'(t) = \sin t + e^t, \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

Stavimo $f(t) \circ\!\!-\!\!\bullet\, F(s)$. Pa vrijedi:

$$f'(t) \circ\!\!-\!\!\bullet\, sF(s) - f(0) = sF(s)$$

$$f''(t) \circ\!\!-\!\!\bullet\, s^2F(s) - sf(0) - f'(0) = s^2F(s).$$

Primjer 31

Riješi diferencijalnu jednadžbu s početnim uvjetom :

$$\begin{cases} f''(t) + f'(t) = \sin t + e^t, \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

Stavimo $f(t) \circ\!\!-\!\!\bullet F(s)$. Pa vrijedi:

$$f'(t) \circ\!\!-\!\!\bullet sF(s) - f(0) = sF(s)$$

$$f''(t) \circ\!\!-\!\!\bullet s^2F(s) - sf(0) - f'(0) = s^2F(s).$$

Znamo i $\sin t + e^t \circ\!\!-\!\!\bullet \frac{1}{s^2+1} + \frac{1}{s-1}$. Stoga vrijedi

$$s^2F(s) + sF(s) = \frac{1}{s^2+1} + \frac{1}{s-1} = \frac{s+s^2}{(s-1)(s^2+1)}$$

odnosno $F(s) = \dfrac{1}{(s-1)(s^2+1)}$.

Inverzna transformacija

Da bismo dobili rješenje diferencijalne jednadžbe još je potrebno odrediti funkciju $f(t)$ za koju vrijedi $f(t) \circ\!\!-\!\!\bullet \frac{1}{(s-1)(s^2+1)}$.

Primjer 32

Odredite funkciju $f(t)$ čiji je transformat $F(s) = \frac{1}{(s-1)(s^2+1)}$. $= \frac{1}{s-1} \cdot \frac{1}{s^2+1}$

$$e^t * \sin t$$

$$f(t) = \cancel{e^t \sin t}$$

Inverzna transformacija

Da bismo dobili rješenje diferencijalne jednadžbe još je potrebno odrediti funkciju $f(t)$ za koju vrijedi $f(t) \circ\!\!-\!\!\!-\!\!\bullet \frac{1}{(s-1)(s^2+1)}$.

Primjer 32

Odredite funkciju $f(t)$ čiji je transformat $F(s) = \frac{1}{(s-1)(s^2+1)}$.

Rj: $\frac{1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\!-\!\!\circ \frac{e^t - \cos t - \sin t}{2} u(t) \neq \cancel{e^t \sin t}$

Da bismo dobili rješenje diferencijalne jednadžbe još je potrebno odrediti funkciju $f(t)$ za koju vrijedi $f(t) \circ\!\!-\!\!\bullet \frac{1}{(s-1)(s^2+1)}$.

Primjer 32

Odredite funkciju $f(t)$ čiji je transformat $F(s) = \frac{1}{(s-1)(s^2+1)}$.

Rj: $\frac{1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\circ \frac{e^t - \cos t - \sin t}{2} u(t)$

Stoga je rješenje prethodne diferencijalne jednadžbe $f(t) = \frac{e^t - \cos t - \sin t}{2} u(t)$.

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$. $= \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \Big/ \cdot (s-1)(s^2+1)$

$s^2+1=0$ nema realnih rješenja

$$3s+1 = A(s^2+1)+(Bs+C)(s-1)$$

$$3s+1 = As^2 + A + Bs^2 + Cs - Bs - C$$

$$0s^2 + 3s + 1 = (A+B)s^2 + (C-B)s + A - C$$

$\langle s^2 \rangle : \quad 0 = A+B$ $\longrightarrow$ $\boxed{B = -A = -2}$

$\langle s \rangle : \quad 3 = C - B$ $\Big| +$ $\rightarrow$ $3 = A + C$ $\quad \boxed{C = 1}$

$\langle 1 \rangle : \quad 1 = A - C$ $\quad +$ $\quad 4 = 2A \Rightarrow \boxed{A = 2}$

$$F(s) = \frac{2}{s-1} + \frac{-2s+1}{s^2+1} = \frac{2}{s-1} - 2\frac{s}{s^2+1} + \frac{1}{s^2+1} \bullet\!\!-\!\!\circ\; 2e^t - 2\cos t + \sin t$$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

(a) $\dfrac{1}{s}$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

$e^{at} f(t) \circ\!\!-\!\!\bullet F(s-a)$ $a=2$

(a) $\frac{1}{s} \bullet\!\!-\!\!\circ u(t)$,

(b) $\frac{s-2}{(s-2)^2+1} \bullet\!\!-\!\!\circ e^{2t}\cos t$ | $\frac{s}{s^2+1} \bullet\!\!-\!\!\circ \cos t$

Primjer 33

Pronađite original funkcije $F(s)=\frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

$e^{at} f(t) \circ\!\!-\!\!\bullet F(s-a)$ $\quad a=-5 \quad F(s+5)$

(a) $\frac{1}{s} \bullet\!\!-\!\!\circ u(t)$,

(b) $\frac{s-2}{(s-2)^2+1} \bullet\!\!-\!\!\circ e^{2t} \cos t\, u(t)$,

(c) $\underbrace{\frac{1}{(s+5)^2+1}}_{F(s+5)} \bullet\!\!-\!\!\circ \underline{e^{-5t} \cdot \sin t}$

$\overbrace{\frac{1}{s^2+1}}^{F(s)} \bullet\!\!-\!\!\circ \overset{f(t)}{\sin t}$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

(a) $\frac{1}{s} \bullet\!\!-\!\!\circ u(t)$,

(b) $\frac{s-2}{(s-2)^2+1} \bullet\!\!-\!\!\circ e^{2t} \cos t\, u(t)$,

(c) $\frac{1}{(s+5)^2+1} \bullet\!\!-\!\!\circ e^{-5t} \sin t\, u(t)$,

(d) $\frac{se^{-3s}}{s^2+1} \bullet\!\!-\!\!\circ \cos(t-3)\,u(t-3)$

$a = 3$

$f(t-a)\,u(t-a) \circ\!\!-\!\!\bullet e^{-as}F(s)$

$F(s) \qquad f(t)$

$\frac{s}{s^2+1} \bullet\!\!-\!\!\circ \cos t$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

$$f(t-a)\,u(t-a) \circ\!\!-\!\!\!-\!\!\bullet e^{-as}F(s)$$

$$e^{at}f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s-a)$$

(a) $\frac{1}{s} \bullet\!\!-\!\!\!-\!\!\circ u(t)$,

(b) $\boxed{\frac{s-2}{(s-2)^2+1}} \bullet\!\!-\!\!\!-\!\!\circ \boxed{e^{2t}\cos t\, u(t)}$,

(c) $\frac{1}{(s+5)^2+1} \bullet\!\!-\!\!\!-\!\!\circ e^{-5t}\sin t\, u(t)$,

(d) $\frac{se^{-3s}}{s^2+1} \bullet\!\!-\!\!\!-\!\!\circ \cos(t-3)\, u(t-3)$,

(e) $\frac{(s-2)e^{-3s}}{(s-2)^2+1}$ $\bullet\!\!-\!\!\!-\!\!\circ\ e^{2(t-3)}\cos(t-3)\,u(t-3)$

$$\frac{s}{s^2+1} \bullet\!\!-\!\!\!-\!\!\circ \cos t$$

Primjer 33

Pronađite original funkcije $F(s) = \frac{3s+1}{(s-1)(s^2+1)}$.

Rj: $\frac{3s+1}{(s-1)(s^2+1)} \bullet\!\!-\!\!\!-\!\!\circ (2e^t - 2\cos t + \sin t)\, u(t)$

Primjer 34

Pronađite originale donjih funkcija:

(a) $\displaystyle \frac{1}{s} \bullet\!\!-\!\!\!-\!\!\circ u(t),$

(b) $\displaystyle \frac{s-2}{(s-2)^2+1} \bullet\!\!-\!\!\!-\!\!\circ e^{2t}\cos t\, u(t),$

(c) $\displaystyle \frac{1}{(s+5)^2+1} \bullet\!\!-\!\!\!-\!\!\circ e^{-5t}\sin t\, u(t),$

(d) $\displaystyle \frac{se^{-3s}}{s^2+1} \bullet\!\!-\!\!\!-\!\!\circ \cos(t-3)\, u(t-3),$

(e) $\displaystyle \frac{(s-2)e^{-3s}}{(s-2)^2+1} \bullet\!\!-\!\!\!-\!\!\circ e^{2(t-3)}\cos(t-3)\, u(t-3).$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

$$(a\pm b)^2 = a^2 \pm 2ab + b^2$$

$$\frac{s}{16s^2-16s+5} = \frac{1}{16}\cdot\frac{s}{\boxed{s^2 - s + \frac{5}{16}}} =$$

nema real. nult.

$$\cancel{s_{1,2} = \frac{-1\pm\sqrt{1-4\cdot\frac{5}{16}}}{2}} \notin \mathbb{R}$$

NEMOGU

$\hookrightarrow \cancel{(s-s_1)(s-s_2)\dots}$

svođenje na potp. kvadr.

$$= \frac{1}{16}\cdot\frac{s}{\underbrace{s^2-2\cdot s\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2}_{\left(s-\frac{1}{2}\right)^2}-\left(\frac{1}{2}\right)^2+\frac{5}{16}} = \frac{1}{16}\cdot\frac{\left(s-\frac{1}{2}\right)+\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{1}{16}} =$$

$$\frac{s}{16s^2-16s+5} = \frac{1}{16}\left[\frac{\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2+\frac{1}{16}} + \frac{\left(1/2\cdot 4\right)\cdot\frac{1}{4}}{\left(s-\frac{1}{2}\right)^2+\frac{1}{16}}\right]$$

$$\frac{1}{16} = \left(\frac{1}{4}\right)^2$$

$$e^{\frac{1}{2}t}\cdot\cos\left(\frac{1}{4}t\right) \qquad 2\cdot e^{\frac{1}{2}t}\cdot\sin\left(\frac{1}{4}t\right)$$

$$\frac{s}{16s^2-16s+5} \circ\!-\!\circ \frac{e^{t/2}}{16}\left[\cos\left(\frac{t}{4}\right)+2\sin\left(\frac{t}{4}\right)\right]u(t)$$

$$\frac{s\,e^{-s}}{16s^2-16s+5} \circ\!-\!\circ \frac{e^{\frac{t-1}{2}}}{16}\left[\cos\left(\frac{t-1}{4}\right)+2\sin\left(\frac{t-1}{4}\right)\right]u(t-1)$$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5}$ $\bullet\!\!-\!\!\!-\!\!\circ$ $\frac{1}{16}e^{\frac{t-1}{2}}\cos\frac{t-1}{4}\,u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}}\sin\frac{t-1}{4}\,u(t-1)$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5}$ ●—○ $\frac{1}{16}e^{\frac{t-1}{2}}\cos\frac{t-1}{4}\,u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}}\sin\frac{t-1}{4}\,u(t-1)$

Primjer 36 (DZ)

za kraj

Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

$\cancel{\frac{s+1}{(s+1)(s+\ldots)}}$ = parc. . . -

s^2+s+1 $D = b^2-4ac = 1-4 = -3 < 0$

nema real. nj.

kao i u Pr 35

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5} \bullet\!\!-\!\!\!-\!\!\circ \frac{1}{16}e^{\frac{t-1}{2}} \cos\frac{t-1}{4}\, u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}} \sin\frac{t-1}{4}\, u(t-1)$

Primjer 36 (DZ)

Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

Rj: $\left[e^{-\frac{t-\pi}{2}} \cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}}e^{-\frac{t-\pi}{2}} \sin\frac{\sqrt{3}}{2}(t-\pi)\right] u(t-\pi)$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5} \bullet\!\!-\!\!\circ \frac{1}{16}e^{\frac{t-1}{2}}\cos\frac{t-1}{4}u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}}\sin\frac{t-1}{4}u(t-1)$

Primjer 36 (DZ)

Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

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Primjer 37 (dio s rastavom na parc. raz. (DZ))

Odredite original od $F(s) = \frac{1}{s^2+\underbrace{4s}_{s+3s}+3}$. $= \frac{1}{(s+1)(s+3)}$

$D = b^2 - 4ac = 16 - 4\cdot 3 = 4 > 0$

$s^2+4s+3=0$

$s_1 = -1$

$s_2 = -3$

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \text{(DZ)} \dots = \frac{1/2}{s+1} + \frac{-1/2}{s+3}$$

riješiti 2x2 sustav

$F(s) \bullet\!\!-\!\!\circ \left(\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)u(t)$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5} \bullet\!\!-\!\!\circ\ \frac{1}{16}e^{\frac{t-1}{2}} \cos\frac{t-1}{4}\, u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}} \sin\frac{t-1}{4}\, u(t-1)$

Primjer 36 (DZ)

Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

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Primjer 37

Odredite original od $F(s) = \frac{1}{s^2+4s+3}$.

Rj: $\frac{e^{-t}-e^{-3t}}{2}$

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5} \bullet\!\!-\!\!\circ \frac{1}{16}e^{\frac{t-1}{2}} \cos \frac{t-1}{4}\, u(t-1) + \frac{1}{8}e^{\frac{t-1}{2}} \sin \frac{t-1}{4}\, u(t-1)$

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Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

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Primjer 37

Odredite original od $F(s) = \frac{1}{s^2+4s+3}$.

Rj: $\frac{e^{-t}-e^{-3t}}{2}$

Primjer 38 (DZ)

Odredite original od $F(s) = \frac{s+1}{s^2(s-1)(s-2)}$. $= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2} \quad / \cdot s^2(s-1)(s-2)$

(DZ)

4 x 4 sustav ...

Primjer 35

Odredite original od $F(s) = \frac{se^{-s}}{16s^2-16s+5}$.

Rj: $\frac{se^{-s}}{16s^2-16s+5} \bullet\!\!-\!\!\circ \frac{1}{16} e^{\frac{t-1}{2}} \cos \frac{t-1}{4}\, u(t-1) + \frac{1}{8} e^{\frac{t-1}{2}} \sin \frac{t-1}{4}\, u(t-1)$

Primjer 36 (DZ)

Odredite original od $F(s) = \frac{(s+1)e^{-\pi s}}{s^2+s+1}$.

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Primjer 37

Odredite original od $F(s) = \frac{1}{s^2+4s+3}$.

Rj: $\frac{e^{-t}-e^{-3t}}{2}$

Primjer 38 (DZ)

Odredite original od $F(s) = \frac{s+1}{s^2(s-1)(s-2)}$.

Rj: $-\frac{3}{4} - \frac{1}{2}t + \frac{2}{3}e^{t} + \frac{1}{12}e^{-2t}$

Teorem o konačnoj i početnoj vrijednosti

Pretpostavimo da $f(t)$ i $f'(t)$ obje imaju Laplaceov transformat definiran za sve $s > 0$. Ako postoje limesi $\lim\limits_{t\to+\infty} f(t) = f(+\infty)$, $\lim\limits_{s\to 0+} sF(s)$ i $\lim\limits_{s\to+\infty} sF(s)$ onda vrijedi

$$f(+\infty) = \boxed{\lim_{s\to 0+} sF(s)} \quad \text{i} \quad \left(f(0) = \boxed{\lim_{s\to+\infty} sF(s)} \right)$$

gdje je $f(t) \circ\!\!-\!\!\bullet\, F(s)$. Štoviše, isto vrijedi i čim $F(s)$ nema polova za $\mathrm{Re}\, s \geqslant 0$.

Izvod na predavanju.

dif jdb.

$y''' + e^{-t} y'' + \dots = \sin t + \dots$

$y(0) =$

$y'(0) = -$

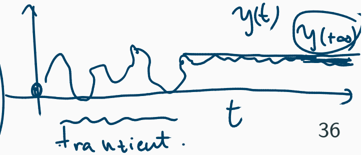
$\mathcal{L}$

$Y(s)[\dots] = [\ ]$

$Y(s) = \boxed{\dfrac{s\dots}{(s+1)(s-3)\dots}}$

NE TREBA

$\circ\!\!-\!\!\circ\ y(t)$??



$$f(t) \circ\!\!-\!\!\bullet F(s)$$

$$f'(t) \circ\!\!-\!\!\bullet sF(s) - f(0)$$

$$\rightarrow \int_0^\infty f'(t)\, \cancel{e^{-st}}\, dt = sF(s) - f(0) \qquad \lim_{s \to 0+}$$

$$\cancel{e^{-st}} \to 1$$

$$s \to +\infty$$

$$\int_0^\infty f'(t)\, dt = \lim_{s \to 0+} sF(s) - f(0)$$

$$\boxed{f(+\infty)} - \cancel{f(0)} = \lim_{s \to 0+} sF(s) - \cancel{f(0)}$$

$$\boxed{f(+\infty) = \lim_{s \to +\infty} sF(s)}$$

$$0 = \int_0^\infty f'(t)\, \cancel{e^{-st}}\, dt = \left(\lim_{s \to +\infty} sF(s)\right) - f(0)$$

$$0$$

$$\boxed{(e^{-\infty} \to 0)}$$

$$0$$

$$f(0) = \lim_{s \to +\infty} sF(s)$$

Teorem o konačnoj i početnoj vrijednosti

Pretpostavimo da $f(t)$ i $f'(t)$ obje imaju Laplaceov transformat definiran za sve $s > 0$. Ako postoje limesi $\lim_{t\to+\infty} f(t) = f(+\infty)$, $\lim_{s\to 0+} sF(s)$ i $\lim_{s\to+\infty} sF(s)$ onda vrijedi

$$f(+\infty) = \lim_{s\to 0+} sF(s) \qquad \text{i} \qquad f(0) = \lim_{s\to+\infty} sF(s)$$

gdje je $f(t) \circ\!\!-\!\!\bullet\, F(s)$. Štoviše, isto vrijedi i čim $F(s)$ nema polova za $\mathrm{Re}\, s \geqslant 0$.

Izvod na predavanju.

Primjer 39

Bez određivanja $f(t)$ odredite $f(0)$ i $f(+\infty)$ pri čemu je $\boxed{f(t) \circ\!\!-\!\!\bullet\, \frac{2s+5}{s(s+7)}}$.

$\lim_{t\to+\infty} f(t)$

1° način $\quad \dfrac{2s+5}{s(s-7)}$ •—○ - - - ?? $= f(t)$ $\quad (\exists z)$ ○ . . .

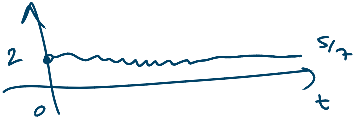
→ rast. na par. razl.

2° način $\quad \cancel{f(t)}$

$$f(+\infty) = \lim_{s\to 0^+} sF(s) = \lim_{s\to 0^+} \cancel{s}\cdot\frac{2s+5}{\cancel{s}(s+7)} = \lim_{s\to 0^+} \frac{2s+5}{s+7} = \boxed{\frac{5}{7}}$$

$$\boxed{f(+\infty) = \frac{5}{7}}$$

$$f(0) = \lim_{s\to +\infty} sF(s) = \lim_{s\to +\infty} \frac{2s+5}{s+7} = \frac{2}{1} = \boxed{2}$$



$$\boxed{f(0)=2}$$

$$\boxed{f(+\infty) = \frac{5}{7}}$$

Teorem o konačnoj i početnoj vrijednosti

Pretpostavimo da $f(t)$ i $f'(t)$ obje imaju Laplaceov transformat definiran za sve $s > 0$. Ako postoje limesi $\lim_{t\to+\infty} f(t) = f(+\infty)$, $\lim_{s\to 0+} sF(s)$ i $\lim_{s\to+\infty} sF(s)$ onda vrijedi

$$f(+\infty) = \lim_{s\to 0+} sF(s) \qquad \text{i} \qquad f(0) = \lim_{s\to+\infty} sF(s)$$

gdje je $f(t) \circ\!\!-\!\!\bullet F(s)$. Štoviše, isto vrijedi i čim $F(s)$ nema polova za $\operatorname{Re} s \geqslant 0$.

Izvod na predavanju.

Primjer 39

Bez određivanja $f(t)$ odredite $f(0)$ i $f(+\infty)$ pri čemu je $f(t) \circ\!\!-\!\!\bullet \frac{2s+5}{s(s+7)}$.
Rj: $f(0) = 2, f(+\infty) = \frac{5}{7}$

Konvolucija

Motivacija

Prisjetimo se: $\mathcal{L}\left\{t\right\} =$

Motivacija

Prisjetimo se: $\mathcal{L}\left\{t\right\} = \frac{1}{s^2}$,
ali pazi! $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\left\{t \cdot t\right\} =$

Motivacija

Prisjetimo se: $\mathcal{L}\left\{t\right\} = \frac{1}{s^2}$,
ali pazi! $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\left\{t \cdot t\right\} = \frac{2}{s^3} \neq \mathcal{L}\left\{t\right\} \cdot \mathcal{L}\left\{t\right\}$.

Motivacija

Prisjetimo se: $\mathcal{L}\{t\} = \frac{1}{s^2}$,

ali pazi! $\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = \frac{2}{s^3} \neq \mathcal{L}\{t\} \cdot \mathcal{L}\{t\}$.

ZAPAMTI!

Ako je $f(t) \circ\!\!-\!\!\bullet F(s)$ i $g(t) \circ\!\!-\!\!\bullet G(s)$ onda NE VRIJEDI pravilo:

$$\cancel{f(t) \cdot g(t) \circ\!\!-\!\!\bullet F(s) \cdot G(s)}$$

Motivacija

Prisjetimo se: $\mathcal{L}\{t\} = \frac{1}{s^2}$,

ali pazi! $\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = \frac{2}{s^3} \neq \mathcal{L}\{t\} \cdot \mathcal{L}\{t\}$.

ZAPAMTI!

Ako je $f(t) \circ\!\!-\!\!\bullet F(s)$ i $g(t) \circ\!\!-\!\!\bullet G(s)$ onda NE VRIJEDI pravilo:

$$\cancel{f(t) \cdot g(t) \circ\!\!-\!\!\bullet F(s) \cdot G(s)}$$

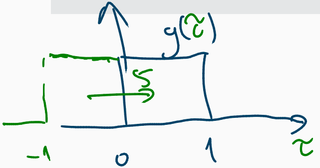
Kojoj operaciji u gornjoj domeni odgovara množenje transformata u donjoj?

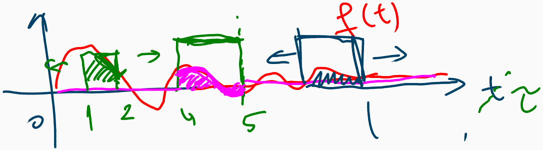
$$f(t) * g(t) \circ\!\!-\!\!\bullet F(s) \cdot G(s)$$

Definicija

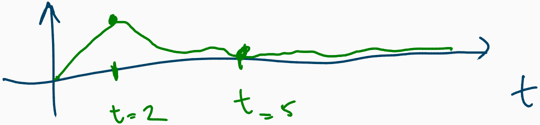
Ako su $f(t)$ i $g(t)$ dvije funkcije za koje vrijedi $f(t) = g(t) = 0$ za $t < 0$ onda definiramo **konvoluciju funkcija** f **i** g u oznaci $f * g$ kao funkciju

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\,.$$





$g(5-\tau)$
$g(-(\tau-5))$



PON UTO SRI

temp = x, x, 0, 1, -3, 15, 17, 20, 15, -5, ... f(t)

Promet u šibenskoj tržnici:

3x prosječna temp u zadnja tri dana

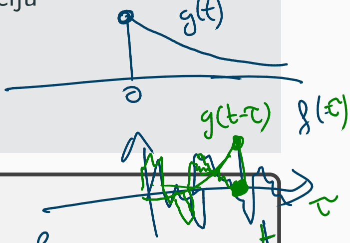
Konvolucija

$\frac{-2}{3}, \frac{13}{3}, \frac{29}{3}, \ldots$

Definicija

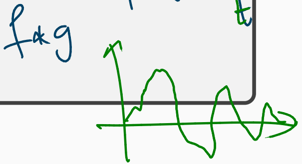
Ako su $f(t)$ i $g(t)$ dvije funkcije za koje vrijedi $f(t) = g(t) = 0$ za $t < 0$ onda definiramo **konvoluciju funkcija f i g** u oznaci $f * g$ kao funkciju

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, \mathrm{d}\tau\,.$$



Ako je $f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s)$ i $g(t) \circ\!\!-\!\!\!-\!\!\bullet G(s)$ onda vrijedi

$$(f * g)(t) \circ\!\!-\!\!\!-\!\!\bullet \boxed{F(s)}G(s).$$

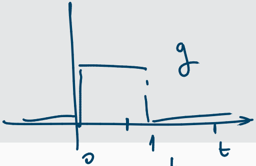


Bez izvoda.

Primjer 40

Neka je $g(t) = g_{[0,1]}(t)$. Odredite $g * g$ direktnim računom. Zatim se uvjerite da

vrijedi $\mathcal{L}\{g * g\} = \mathcal{L}\{g\}\,\mathcal{L}\{g\}$.



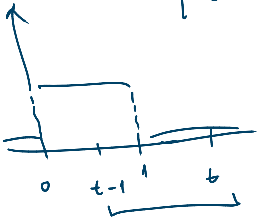
$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau\,.$$

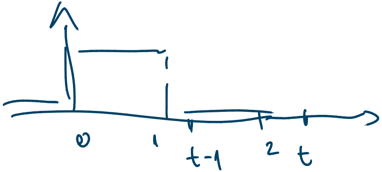
$$(g*g)(t) = \int_0^t g(u)\cdot g(t-u)\,du = \begin{cases} 0\le t\le 1: & \int_0^t 1\cdot g(t-u)\,du = \\ t>1: & \int_0^1 1\cdot g(t-u)\,du = \end{cases} = \left[\begin{array}{l} t-u=v \\ -du=dv \end{array}\right]$$

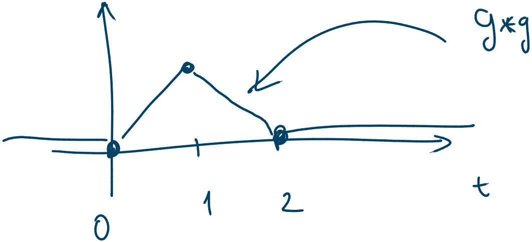
u	v
0	t
t	0
1	t-1

$$= \begin{cases} t\le 1: & \int_t^0 g(v)(-dv) \\ t>1: & \int_t^{t-1} g(v)(-dv) \end{cases} =$$

$$= \begin{cases} t \le 1 : & \int_0^t \underbrace{g(v)}_{1} dv \\ t > 1 : & \int_{t-1}^t g(v)\, dv \end{cases} = \begin{cases} t \le 1 : & \int_0^t 1\, dv = \underline{\underline{t}} \\ 1 < t \le 2 : & \int_{t-1}^1 1\, dv = 2 - t \\ t > 2 : & \int_{t-1}^t 0\, dv = 0 \end{cases}$$







Primjer 40

Neka je $g(t) = g_{[0,1]}(t)$. Odredite $g * g$ direktnim računom. Zatim se uvjerite da vrijedi $\mathcal{L}\{g * g\} = \mathcal{L}\{g\}\mathcal{L}\{g\}$. **Rj:** $(g_{[0,1]} * g_{[0,1]})(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2 - t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{inače.} \end{cases}$

https://upload.wikimedia.org/wikipedia/commons/6/6a/Convolution_of_box_signal_with_itself2.gif

Primjer 40

Neka je $g(t) = g_{[0,1]}(t)$. Odredite $g * g$ direktnim računom. Zatim se uvjerite da vrijedi $\mathcal{L}\{g * g\} = \mathcal{L}\{g\}\mathcal{L}\{g\}$. **Rj:** $(g_{[0,1]} * g_{[0,1]})(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2 - t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{inače.} \end{cases}$

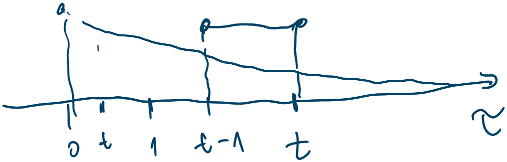
https://upload.wikimedia.org/wikipedia/commons/6/6a/Convolution_of_box_signal_with_itself2.gif

Primjer 41

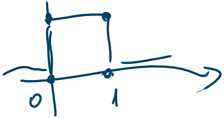
Neka je $f(t) = e^{-t}u(t)$ i $g(t) = g_{[0,1]}(t)$. Odredite $f * g$ direktnim računom. Zatim se uvjerite da vrijedi $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

$\| \quad g * f$

$$(f*g)(t) = \int_0^t e^{-\tau} \cdot g_{[0,1]}(t-\tau)\, d\tau = \left[\begin{array}{ll} u = t - \tau & \\ \frac{du}{d\tau} = -1 & du = -d\tau \end{array} \right]$$



$$= \int_t^0 e^{u-t}\, g_{[0,1]}(u)\,(-du) =$$



$$= \int_0^t e^{u-t}\, g_{[0,1]}(u)\, du =$$

$$= \begin{cases} 0 \le t \le 1 : & \int_0^t e^{u-t}\,du = e^{-t}\left(e^u\right)\Big|_0^t = e^{-t}\left(e^t - 1\right) = 1 - e^{-t} \\ t > 1 : & \int_0^1 e^{u-t}\,du = e^{-t}\left(e^u\right)\Big|_0^1 = e^{-t}(e-1) = e^{1-t} - e^{-t} \end{cases}$$

Primjer 40

Neka je $g(t) = g_{[0,1]}(t)$. Odredite $g * g$ direktnim računom. Zatim se uvjerite da vrijedi $\mathcal{L}\{g * g\} = \mathcal{L}\{g\}\mathcal{L}\{g\}$. **Rj:** $(g_{[0,1]} * g_{[0,1]})(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2 - t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{inače.} \end{cases}$

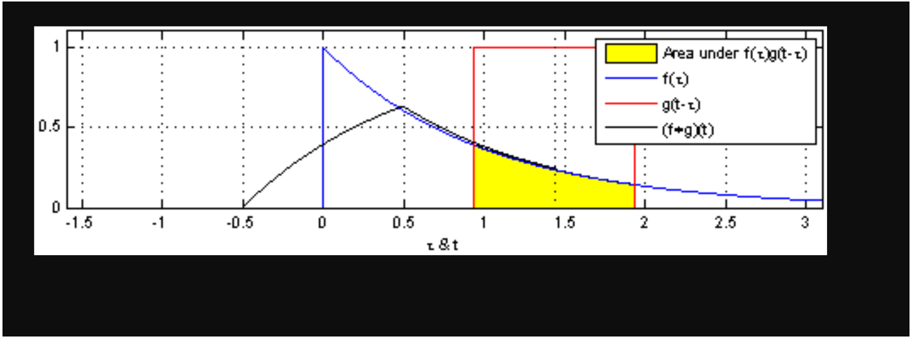
https://upload.wikimedia.org/wikipedia/commons/6/6a/Convolution_of_box_signal_with_itself2.gif

Primjer 41

Neka je $f(t) = e^{-t}u(t)$ i $g(t) = g_{[0,1]}(t)$. Odredite $f * g$ direktnim računom. Zatim se uvjerite da vrijedi $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

Rj: $(f * g)(t) = \begin{cases} 0, & t < 0, \\ 1 - e^{-t}, & 0 \leqslant t \leqslant 1 \\ (e-1)e^{-t}, & t > 1. \end{cases}$

https://upload.wikimedia.org/wikipedia/commons/b/b9/Convolution_of_spiky_function_with_box2.gif



nastavak

g^{15}

Neka je $g(t) = g_{[0,1]}(t)$. Odredite $g * g$ direktnim računom. Zatim se uvjerite da vrijedi $\boxed{\mathcal{L}\{g * g\} = \mathcal{L}\{g\}\,\mathcal{L}\{g\}}$. **Rj:** $(g_{[0,1]} * g_{[0,1]})(t) = \boxed{\begin{cases} t, & 0 \leqslant t \leqslant 1 \\ 2-t, & 1 \leqslant t \leqslant 2 \\ 0, & \text{inače.} \end{cases}}$

$$g*g \overset{?}{\circ\!\!-\!\!\bullet} \frac{1-e^{-s}}{s} \cdot \frac{1-e^{-s}}{s} = \frac{1-2e^{-s}+e^{-2s}}{s^2}$$

invertati

$$g_{[a,b]} = u(t-a) - u(t-b) \circ\!\!-\!\!\bullet \frac{e^{-as}-e^{-bs}}{s}$$

$$g_{[0,1]} \circ\!\!-\!\!\bullet \frac{1-e^{-s}}{s}$$

$$g*g = t \cdot g_{[0,1]} + (2-t)\cdot g_{[1,2]} =$$

$$= t\big(u(t)-u(t-1)\big) + (2-t)\big(u(t-1)-u(t-2)\big)$$

$$= t\,u(t) - t\,u(t-1) + (2-t)\,u(t-1) + (t-2)\,u(t-2)$$

$$= \boxed{t\,u(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2)} \circ\!\!-\!\!\bullet \boxed{\frac{1}{s^2} - 2\cdot e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}}$$

$$f(\overbrace{t-a}^{\tau})\,u(\overbrace{t-a}^{\tau}) \circ\!\!-\!\!\bullet\; e^{-as}F(s)$$

$$t\,u(t) \circ\!\!-\!\!\bullet\; \frac{1}{s^2}$$

$$(\overbrace{t-1}^{\tau})\,u(\overbrace{t-1}^{\tau}) = \tau\,u(\tau) \circ\!\!-\!\!\bullet\; e^{-s}\cdot\frac{1}{s^2}$$

$$(\overbrace{t-2}^{\tau})\,u(\overbrace{t-2}^{\tau}) = \tau\,u(\tau) \circ\!\!-\!\!\bullet\; e^{-2s}\,\frac{1}{s^2}$$

$$
\begin{tikzcd}
f \arrow[d] & * & g \arrow[d] & \checkmark \\
F & \cdot & G \arrow[r, phantom] & \arrow[u, "??\,\text{inventirati}\ \mathcal{L}^{-1}"']
\end{tikzcd}
$$

$$D = 0^2 - 4\cdot 1\cdot 1 = -4 < 0$$

Primjer 42

$$s^2 + 0s + 1 = 0$$

Odredite original transformata $F(s) = \dfrac{s}{(s^2+1)^2}$.

$$F(s) = \frac{s}{(s^2+1)} \cdot \frac{1}{(s^2+1)} \quad \bullet\!\!-\!\!\circ \quad \cos t * \sin t = \int_0^t \cos\tau \sin(t-\tau)\, d\tau$$

$$\frac{s}{s^2+1} \bullet\!\!-\!\!\circ \cos t \qquad \frac{1}{s^2+1} \bullet\!\!-\!\!\circ \sin t$$

$$= \int_0^t \cos\tau \left(\sin t \cos\tau - \cos t \sin\tau\right) d\tau =$$

$$= \int_0^t \sin t \cos^2\tau - \int_0^t \cos t \sin\tau \cos\tau \, d\tau =$$

$$= \sin t \int_0^t \frac{1+\cos 2\tau}{2} d\tau - \cos t \int_0^t \frac{\sin 2\tau}{2} dt$$

$$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$$

$$= \sin t \left(\frac{\tau}{2} + \frac{1}{4}\sin 2\tau \right)\Big|_0^t + \frac{\cos t}{4} (\cos 2\tau)\Big|_0^t =$$

$$\sin 2x = 2 \sin x \cos x$$

$$= \underbrace{\sin t \left(\frac{t}{2} + \frac{\sin 2t}{4} \right) + \frac{\cos t}{4} (\cos 2t - 1)}_{R} \overset{(\text{Dt} \dots)}{=} \frac{t \sin t}{2}$$

Alternativno $\cancel{(D2^*)}$

$$\sin t \circ\!\!-\!\!\bullet \frac{1}{s^2+1} \xleftarrow{-\frac{d}{ds}}$$

$$t f(t) \circ\!\!-\!\!\bullet -F'(s)$$ ✓

$$t \sin t \circ\!\!-\!\!\bullet \frac{2s}{(s^2+1)^2} \Rightarrow \frac{t \sin t}{2} \circ\!\!-\!\!\bullet \frac{s}{(s^2+1)^2}$$

Primjer 42

Odredite original transformata $F(s) = \dfrac{s}{(s^2+1)^2}$.

Rj: $\dfrac{s}{(s^2+1)^2} = \dfrac{s}{s^2+1} \cdot \dfrac{1}{s^2+1} \bullet\!\!-\!\!\circ (\sin * \cos)(t) = \frac{1}{2} t \sin t \, u(t)$ ✓

Primjer 43

Izračunajte $e^t * e^t$ i direktno i preko Laplaceovih transformata.

1° način $\quad \dfrac{1}{s-1} \cdot \dfrac{1}{s-1} = \dfrac{1}{(s-1)^2} \overset{??}{\bullet\!\!-\!\!\circ} \underline{\underline{te^t}}$

$$\frac{1}{(\underbrace{s-1}_{k})^2} \bullet\!\!-\!\!\circ\ e^t\, t\, u(\cancel{t})$$

$$e^{at} f(t) \circ\!\!-\!\!\bullet F(\overbrace{s-a}^{k})$$

$$\frac{1}{s^2} \bullet\!\!-\!\!\circ\ t\, u(\cancel{t})$$

$$t f(t) \circ\!\!-\!\!\bullet -F'(s)$$

1º B

$$-\frac{d}{ds}$$

$$\frac{1}{s-1} \bullet\!\!-\!\!\circ\ e^t$$

$$\frac{1}{(s-1)^2} \bullet\!\!-\!\!\circ\ t \cdot e^t$$ ✓

2º diretamente:

$$e^t * e^t = \int_0^t e^\tau \cdot e^{t-\tau}\, d\tau = \int_0^t e^t\, d\tau = e^t \int_0^t d\tau = e^t \tau\Big|_0^t = e^t \cdot t$$ ✓

Primjer 42

Odredite original transformata $F(s) = \dfrac{s}{(s^2+1)^2}$.

Rj: $\dfrac{s}{(s^2+1)^2} = \dfrac{s}{s^2+1} \cdot \dfrac{1}{s^2+1} \bullet\!\!-\!\!\!-\!\!\circ (\sin * \cos)(t) = \frac{1}{2} t \sin t\, u(t)$

Primjer 43

Izračunajte $e^t * e^t$ i direktno i preko Laplaceovih transformata. **Rj:** $te^t\, u(t)$

Rješavanje diferencijalnih i integralnih jednadžbi

Tehnika Laplaceove transformacije pomaže nam rješiti:

- Linearne diferencijalne jednadžbe s konstantnim koeficijentima

$$\text{npr.}\qquad \begin{cases} y''(t) + y(t) = \sin(t)u(t) \\ y(0) = 1,\ y'(0) = 2 \end{cases}$$

Tehnika Laplaceove transformacije pomaže nam riješiti:

- Linearne diferencijalne jednadžbe s konstantnim koeficijentima

npr. $$\begin{cases} y''(t) + y(t) = \sin(t)u(t) \\ y(0) = 1,\ y'(0) = 2 \end{cases}$$

- Linearne integralne jednadžbe konvolucijskog tipa

sin t * y = y * sin t

npr. $$\underline{y(t)} = t^2 + \underbrace{\int_0^t \sin\tau\, y(t-\tau)\,\mathrm{d}\tau}$$

Tehnika Laplaceove transformacije pomaže nam rješiti:

- Linearne diferencijalne jednadžbe s konstantnim koeficijentima

$$\text{npr.} \qquad \begin{cases} y''(t) + y(t) = \sin(t)u(t) \\ y(0) = 1,\, y'(0) = 2 \end{cases}$$

- Linearne integralne jednadžbe konvolucijskog tipa

$$\text{npr.} \qquad y(t) = t^2 + \int_0^t \sin\tau\, y(t-\tau)\,\mathrm{d}\tau$$

- Integro-diferencijalne jednadžbe

$$\text{npr.} \qquad \begin{cases} y'(t) - y(t) + \int_0^t y(\tau)e^{-t}\,\mathrm{d}\tau \ = \cos t\, u(t) \\ y(0) = 3 \end{cases}$$

Linearne diferencijalne jednadžbe s konstantnim koeficijentima

Primjer 44

Riješite jednadžbu:

output

input

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = 0,\ y'(0) = 0 \end{cases} \quad \Big/ \mathcal{L}$$

pri čemu je funkcija $f(t) = u(t)$. $\circ\!\!-\!\!\bullet\ F(s) = \frac{1}{s}$

? $y(t) \circ\!\!-\!\!\bullet\ Y(s)$

$y''(t) \circ\!\!-\!\!\bullet\ s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$s^2Y(s) + Y(s) = F(s)$

$Y(s)\left[s^2+1\right] = F(s)$

$\boxed{Y(s) = \frac{1}{s^2+1} \cdot F(s)} = \frac{1}{(s^2+1)s}$

$$Y(s) = \frac{1}{(s^2+1)s} = \frac{As+B}{s^2+1} + \frac{C}{s} \quad \Big| \cdot (s^2+1)\,s$$

$$1 = s(As+B) + C(s^2+1)$$

$$1 = (A+C)s^2 + Bs + C$$

$\langle 1 \rangle : \boxed{1 = C}$

$\langle s \rangle : \boxed{0 = B}$

$\langle s^2 \rangle : 0 = A + C \Rightarrow \boxed{A = -1}$

$$Y(s) = \frac{-s}{s^2+1} + \frac{1}{s} \;\bullet\!\!-\!\!\circ\; \underline{\left(-\cos t + 1\right)u(t)} \rightarrow y(t) \;//$$

$f(t)$ input $\longrightarrow$ $\boxed{\text{dif. jdb.}}$ $\xrightarrow{y(t)}$ output

$\big\downarrow \mathcal{L}$

$F(s) \longrightarrow \boxed{\times \left(\frac{1}{s^2+1}\right)} \xrightarrow{Y(s)}$

output $y(t) = \boxed{sys(t)} * f(t)$ input

$$Y(s) = \frac{1}{s^2+1} \bullet F(s) \quad \big\uparrow \mathcal{L}^{-1}$$

impulsni odziv sustav

$(sys(t) = \sin(t))$

Primjer 44

Riješite jednadžbu:

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = 0,\; y'(0) = 0 \end{cases}$$

pri čemu je funkcija $f(t) = u(t)$. **Rj:** $y(t) = (1 - \cos t)\, u(t)$

Primjer 45 (*)

Ako je $f(t)$ proizvoljna funkcija, kako bismo zapisali opće rješenje gornje diferencijalne jednadžbe?

Primjer 44

Riješite jednadžbu:

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

pri čemu je funkcija $f(t) = u(t)$. **Rj:** $y(t) = (1 - \cos t)\, u(t)$

Primjer 45 (*)

Ako je $f(t)$ proizvoljna funkcija, kako bismo zapisali opće rješenje gornje diferencijalne jednadžbe? **Rj:** $\underline{y(t)} = \underbrace{f(t) * \sin t} = \int_0^t \underbrace{f(\tau)} \sin(\tau - t)\, \mathrm{d}\tau$

Primjer 46

Riješite jednadžbu:

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = 1,\ y'(0) = 2 \end{cases} \quad \Big/ \mathcal{L}$$

pri čemu je funkcija $f(t) = u(t)$. $\circ\!\!-\!\!\bullet\ \frac{1}{s}$

$y(t) \circ\!\!-\!\!\bullet\ Y(s)$

$y''(t) \circ\!\!-\!\!\bullet\ s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s) - s - 2$

$$s^2 Y(s) - s - 2 + Y(s) = \frac{1}{s}$$

$$Y(s)\left[s^2+1\right] = \frac{1+s^2+2s}{s}$$

$$Y(s) = \frac{1+2s+s^2}{s(s^2+1)} \quad \overset{\mathcal{L}^{-1}}{\bullet\!\!-\!\!\circ} \quad y(t)$$

$$Y(s) = \frac{1+2s+s^2}{s\,(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad \Big| \cdot s(s^2+1)$$

$$1+2s+s^2 = As^2 + A + Bs^2 + Cs$$

$$1+2s+s^2 = (A+B)s^2 + Cs + A$$

$$(1): \quad \boxed{1 = A}$$

$$(s): \quad \boxed{2 = C}$$

$$(s^2): \quad 1 = A+B \Rightarrow \boxed{B=0}$$

$$Y(s) = \frac{1}{s} + \cancel{0\,\frac{s}{s^2+1}} + 2\,\frac{1}{s^2+1}$$

$$\bullet\!\!-\!\!\circ$$

$$y(t) = \underline{\left(1 + 2\sin t\right) u(t)}$$

Primjer 46

Riješite jednadžbu:

$$\begin{cases} y''(t) + y(t) = f(t) \\ y(0) = 1,\, y'(0) = 2 \end{cases}$$

pri čemu je funkcija $f(t) = u(t)$. **Rj:** $y(t) = (1 + 2\sin t)\, u(t)$ //

Primjer 47

Riješite jednadžbu:

$(\sin t * y(t))$

$$y(t) = t^2 + \int_0^t \sin \overset{\tau}{\tau}\, y(\overset{t-\tau}{t-\tau})\, \mathrm{d}\tau \,. \quad / \mathcal{L}$$

$y(t) \circ\!\!-\!\!\bullet Y(s)$

$$Y(s) = \frac{2}{s^3} + \frac{1}{s^2+1} \cdot Y(s)$$

$$Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{2}{s^3}$$

$$Y(s)\,\frac{s^2+\cancel{1}-\cancel{1}}{s^2+1} = \frac{2}{s^3} \quad \Big| \cdot \frac{s^2+1}{s^2}$$

$$Y(s) = \frac{2(s^2+1)}{s^3\cdot s^2} = 2\,\frac{s^2+1}{s^5} = 2\cdot\frac{1}{s^3}+2\,\frac{1}{s^5}$$

$$Y(s) = 2\,\frac{1}{s^3} + \frac{2}{4!}\,\frac{1\cdot 4!}{s^5} \quad \bullet\!\!-\!\!\circ \quad y(t) = \left(t^2 + \frac{\cancel{2}^{\,1}}{\cancel{4!}_{\,12}}\cdot t^4\right)u(t) = \underline{\underline{\left(t^2+\frac{t^4}{12}\right)u(t)}}\ //$$

Nastavak u $11^{\underline{\underline{15}}}$

Primjer 47

Riješite jednadžbu:

$$y(t) = t^2 + \int_0^t \sin\tau \, y(t-\tau) \, \mathrm{d}\tau \, .$$

Rj: $y(t) = \left(t^2 + \frac{t^4}{12}\right) u(t)$ //

Primjer 48

Riješite jednadžbu:

$y(t) * \sin(t)$

$$\begin{cases} y'(t) - y(t) + \overbrace{\int_0^t y(\tau)\,\sin(t-\tau)\,\mathrm{d}\tau}^{} = \cos t \\ y(0) = 0. \end{cases} \quad / \mathcal{L}$$

$y(t) \circ\!\!-\!\!\bullet\ Y(s)$

$y'(t) \circ\!\!-\!\!\bullet\ sY(s) - \cancel{y(0)}$

$$sY(s) - Y(s) + Y(s) \cdot \frac{1}{s^2+1} = \frac{s}{s^2+1}$$

$$Y(s)\left[s-1+\frac{1}{s^2+1}\right]=\frac{s}{s^2+1}$$

$$Y(s)\cdot\frac{s^3+s-s^2-\cancel{1}+\cancel{1}}{\cancel{s^2+1}}=\frac{s}{\cancel{s^2+1}} \quad \Big|\cdot(s^2+1)$$

$$Y(s)=\frac{s \quad |:s}{s^3-s^2+s \quad |:s}$$

$$\boxed{Y(s)=\frac{1}{s^2-s+1}}=\frac{1}{\underbrace{s^2-2\cdot s\cdot\frac{1}{2}+\frac{1}{4}}-\frac{1}{4}+1}=\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$D=(-1)^2-4\cdot1\cdot1=\boxed{-3<0}$$

$$Y(s) = \underbrace{\frac{1}{\underbrace{\left(s-\frac{1}{2}\right)^2}_{K} + \frac{3}{4}}}$$

$$\bullet\!\!-\!\!\circ$$

$$y(t) = \underbrace{\frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) e^{\frac{1}{2}t}} \cdot u(t) \checkmark$$

$$\frac{1}{\frac{\sqrt{3}}{2}} \cdot \frac{1 \cdot \frac{\sqrt{3}}{2}}{s^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \bullet\!\!-\!\!\circ \underbrace{\frac{2}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)}$$

$$\| $$

$$\left(\frac{1}{s^2 + \frac{3}{4}}\right)$$

$$f(at) \circ\!\!-\!\!\bullet \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$f(t-a)\, u(t-a) \circ\!\!-\!\!\bullet e^{-as} F(s)$$

$$e^{at} f(t) \circ\!\!-\!\!\bullet F(\overbrace{s-a}^{K})$$

Primjer 48

Riješite jednadžbu:

$$\begin{cases} y'(t) - y(t) + \displaystyle\int_0^t y(\tau)\,\sin(t-\tau)\,\mathrm{d}\tau = \cos t \\ y(0) = 0. \end{cases}$$

Rj: $y(t) = \dfrac{2}{\sqrt{3}} e^{t/2} \sin\left(\dfrac{\sqrt{3}}{2} t\right) u(t)$

Literatura

Elezović, *Fourierov red i integral. Laplaceova transformacija.* Element, Zagreb